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தொலைநிலைதொடர் கல்வி இயக்ககம்

DIRECTORATE OF DISTANCE AND CONTINUING EDUCATION

B.Sc. (Allied Mathematics)

I YEAR

ALGEBRA AND DIFFERENTIAL EQUATIONS

Sub. Code: JEMA11 **(For Private Circulation only)**

B.Sc. (Allied Mathematics) –I YEAR

JEMA11 - ALGEBRA AND DIFFERENTIAL EQUATIONS

SYLLABUS

Unit-I:

Theory of Equations – Formation of Equations – Relation between roots and coefficients – Reciprocal equations.

Unit-II:

Transformation of Equations – Approximate solutions to equations – Newton's method and Horner's method.

Unit-III:

Matrices – Characteristic equation of a matrix – Eigen values and Eigen vectors – Cayley Hamilton theorem and simple problems.

Unit-IV:

Differential equation of first order but of higher degree – Equations solvable for p, x, y – Partial differential equations – formations – solutions – Standard form $Pp + Qq = R$.

Unit-V:

Laplace transformation – Inverse Laplace transform.

Recommended Text:

1. Dr.S. Arumugam & Isaac – Allied Mathematics Paper – I, New Gamma Publishing House (2012), Palayamkottai.

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CONTENTS

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Unit-I:

Theory of Equations – Formation of Equations – Relation between roots and coefficients – Reciprocal equations.

1.1 THEORY OF EQUATIONS

Introduction:

In this chapter, we will study about polynomial function and various methods to find out the roots of polynomial equations. 'Solving equations' was an important problem from the beginning of study of Mathematics itself. The notation of complex numbers was first introduced because equation like $x^2 + 1 = 0$ has no solution in the set of real numbers. The "fundamental theorem of algrbra" which states that every polynomial of degree ≥ 1 has atleast one zero was first proved by the famous German Mathematician Karl Fredrich Gauss. We shall look at polynomials in detail and will discuss various methods for solving polynomial equations.

1.2. FORMATION OF EQUATIONS:

Definition-1.2.1:

A function defined by $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$, where $a_0 \neq 0$ is a non-negative integer and $a_i(i = 0,1,2, \dots, n)$ are fixed complex numbers is called a **polynomial of degree n** in x. Then numbers $a_0, a_1, \dots, a_n$ are called **coefficients of f**.

If α is a complex number such that $f(\alpha) = 0$, then α is called **zero** of the polynomials.

Theorem: 1.2.2 (Fundamental Theorem of Algebra):

Every polynomial function of degree $n \geq 1$ has at least one zero.

Remark: Fundamental Theorem of Algebra says that if $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$, where $a_0 \neq 0$ is the given polynomial of degree $n \geq 1$, then there exists a complex number α such that $a_0\alpha^n + a_1\alpha^{n-1} + \cdots + \alpha_n = 0$.

We use the Fundamental Theorem of Algebra, to prove the following result

Theorem: 1.2.3:

Every polynomial function of degree n has n and only n zeros.

Proof:

Let $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$, where $a_0 \neq 0$ be a polynomial of degree $n \geq 1$.

By fundamental theorem of Algebra, $f(x)$ has at least one zero, let α_1 be that zero.

Then $(x - \alpha_1)$ is a factor of $f(x)$.

Therefore, we can write:

$f(x) = (x - \alpha_1)Q_1(x)$, where $Q_1(x)$ is a polynomial function of degree $n - 1$.

If $n - 1 \geq 1$, again by fundamental theorem of Algebra, $Q_1(x)$ has at least one zero, say α_2.

Therefore,

$f(x) = (x - \alpha_1)(x - \alpha_2)Q_2(x)$, where $Q_2(x)$ is a polynomial function of degree $n - 2$.

Repeating the above arguments, we get

$f(x) = (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)Q_n(x)$, where $Q_n(x)$ is a polynomial function of degree $n - n = 0$, i.e., $Q_n(x)$ is a constant.

Equating the coefficient of x^n on both sides of the above equation, we get $Q_n(x) = a_0$.

Therefore,

$$f(x) = a_0(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n).$$

If α is any number other than $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, then $f(x) \neq 0 \Rightarrow \alpha$ is not a zero of $f(x)$.

Hence $f(x)$ has n and only n zeros, namely $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$.

Therefore, every polynomial equation of degree n has n and only n roots.

SOLVED PROBLEMS:

Problem 1. Form the equation with rational coefficients one of whose roots is $\sqrt{2}+\sqrt{3}$.

Solution:

Let $x = \sqrt{2}+\sqrt{3}$.

Therefore, $x-\sqrt{2} = \sqrt{3}$.

Squaring both sides, we get, $x^2 - 2\sqrt{2x} + 2 = 3$.

$\therefore x^2 - 1 = 2\sqrt{2x}$.

Squaring both sides again, we get, $x^4 - 2x^2 + 1 = 8x^2$.

$\therefore x^4 - 10x^2 + 1 = 0$, which is the required equation.

Problem 2. Solve the equation $x^4 - 14x^3 + 46x^2 - 42x + 9 = 0$ given that $5-\sqrt{22}$ is a root.

Solution:

Let $f(x) = x^4 - 14x^3 + 46x^2 - 42x + 9$.

Since $5-\sqrt{22}$ is a root, $5+\sqrt{22}$ is also a root.

$\therefore x-5-\sqrt{22}$ and $x-5+\sqrt{22}$ are factors of $f(x)$.

$\therefore (x-5-\sqrt{22})(x-5+\sqrt{22})$ is a factor of $f(x)$.

(ie) $(x-5)^2 - 22$ is a factor of $f(x)$

(ie) $x^2 - 10x + 3$ is a factor of $f(x)$

By actual division, we get,

$f(x) = (x^2 - 10x + 3)(x^2 - 4x + 3)$ (verify)

$\quad = (x^2 - 10x + 3)(x-3)(x-1)$

$\therefore$ The four roots of $f(x) = 0$ are $5+\sqrt{22}$; $5-\sqrt{22}$; 3 and 1.

Problem 3. If one root of the equation $2x^3 - 11x^2 + 38x - 39 = 0$ is $2 - 3i$. Solve the equation.

Solution:

Let $f(x) = 2x^3 - 11x^2 + 38x - 39$.

Since $2 - 3i$ is a root of $f(x) = 0$, we have $2 + 3i$ is also a root of $f(x) = 0$

$\therefore x - (2 - 3i)$ and $x - (2 + 3i)$ are two factors of $f(x)$.

$\therefore [(x - 2) + 3i][(x - 2) - 3i]$ is a factor of $f(x)$.

(ie) $(x - 2)^2 + 9$ is a factor of $f(x)$

(ie) $x^2 - 4x + 13$ is a factor of $f(x)$

The remaining factor is got by actual division as $2x - 3$

$\therefore$ The roots of are $2 - 3i$; $2 + 3i$ and $\frac{3}{2}$.

Problem 4. Solve $x^4 + 2x^2 - 16x + 77 = 0$ given that one root is $-2 + i\sqrt{7}$.

Solution:

Let $f(x) = x^4 + 2x^2 - 16x + 77$.

Since $-2 + i\sqrt{7}$ is a root, $-2 - i\sqrt{7}$ is also a root.

$\therefore (x + 2 - i\sqrt{7})$ and $(x + 2 + i\sqrt{7})$ are factors of $f(x)$.

$\therefore (x + 2 - i\sqrt{7})(x + 2 + i\sqrt{7})$ is a factor of $f(x)$.

(ie) $(x + 2)^2 + 7$ is a factor of $f(x)$

(ie) $x^2 + 4x + 11$ is a factor of $f(x)$

By actual division, we get,

$f(x) = (x^2 + 4x + 11)(x^2 - 4x + 7)$ (verify)

$= (x^2 + 4x + 11)(x - 2 - i\sqrt{3})(x - 2 + i\sqrt{3})$

$\therefore$ The roots are $-2 + i\sqrt{7}$; $-2 - i\sqrt{7}$; $2 - i\sqrt{3}$ and $2 + i\sqrt{3}$.

EXERCISES:

1. Form the equation of the lowest degree with rational coefficients whose roots are

(i) $1+\sqrt{2}$ and 3.

(ii) $3+\sqrt{5}$ and 1.

(iii) $4\sqrt{3}$ and $5+2i$.

(iv) $\sqrt{3}+i\sqrt{2}$.

2. Solve the equation $x^5-x^4+8x^2-9x-15=0$ if $\sqrt{3}$ and $1-2i$ are two of its roots.

3. Solve the equation $x^3-11x^2+37x-35=0$ given that $3+\sqrt{2}$ is a root.

4. Solve the equation $3x^3-23x^2+72x-70=0$ given that $3+\sqrt{-5}$ is one root.

5. Solve $x^4-x^3-2x^2-7x+3=0$ given that $\frac{1}{2}(3-\sqrt{5})$ is a root.

6. Solve $x^4-4x^2+8x+35=0$ given that $2-\sqrt{3}i$ is a root.

1.3. RELATION BETWEEN ROOTS AND COEFFICIENTS:

Let $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ *be n roots of the* n^{th} *degree equation*

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$$

Then $a_0x^n + a_1x^{n-1} + \cdots + a_n = a_0(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_{n-1})(x-\alpha_n)$

$$= a_0[x^n - x^{n-1}(\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n) + x^{n-2}(\alpha_1\alpha_2+\alpha_2\alpha_3+$$
$$\alpha_3\alpha_4+\cdots) - x^{n-3}(\alpha_1\alpha_2\alpha_3+\alpha_2\alpha_3\alpha_4+\alpha_2\alpha_3\alpha_4+\cdots)+\cdots+(-1)^n\alpha_1\alpha_2\,\alpha_3\ldots\,\alpha_n]$$

Equating coefficients of corresponding power of x on both sides, we get,

$$(-1)^1 a_0 \sum \alpha_1 = a_1$$

$$(-1)^2 a_0 \sum \alpha_1\alpha_2 = a_2$$

$$(-1)^3 a_0 \sum \alpha_1\alpha_2\alpha_3 = a_3$$

…………………………

…………………………

$(-1)^n a_0 \sum \alpha_1\alpha_2\alpha_3 \ldots \alpha_n = a_n$

Rewriting the above relations, we have

$$S_1 = \sum \alpha_1 = (-1)^1 \left(\frac{a_1}{a_0}\right)$$

$$S_2 = \sum \alpha_1\alpha_2 = (-1)^2 \left(\frac{a_2}{a_0}\right)$$

$$S_3 = \sum \alpha_1\alpha_2\alpha_3 = (-1)^3 \left(\frac{a_3}{a_0}\right)$$

……………………………..

……………………………..

$$S_n = \sum \alpha_1\alpha_2\alpha_3 \ldots \alpha_n = (-1)^n \left(\frac{a_n}{a_0}\right)$$

where S_i denotes the sum of the products of roots taken i at a time.

In particular, if a **quadratic equation**

$ax^2 + bx + c = 0$ has roots α and β.

Then $S_1 = \alpha + \beta = -\frac{b}{a}$ and $S_2 = \alpha\beta = \frac{c}{a}$.

If a **cubic equation**

$ax^3 + bx^2 + cx + d = 0$ has roots α, β and γ

Then

$$S_1 = \alpha + \beta + \gamma = -\frac{b}{a}$$

$$S_2 = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$

$$S_3 = \alpha\beta\gamma = -\frac{d}{a}.$$

SOLVED PROBLEMS:

Problem 1.

If $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ are the roots of the equation $x^n - p_1 x^{n-1} + p_2 x^{n-2} - \cdots (-1)^n p_n = 0$, find the value of $(1+\alpha_1)(1+\alpha_2) \dots (1+\alpha_n)$.

Solution:

For the given equation, we have,

$$\sum \alpha_1 = p_1$$

$$\sum \alpha_1 \alpha_2 = p_2$$

$$\sum \alpha_1 \alpha_2 \alpha_3 = p_3$$

……………………………..

……………………………..

$$\alpha_1 \alpha_2 \alpha_3 \dots \alpha_n = p_n$$

Now,

$$(1+\alpha_1)(1+\alpha_2) \dots (1+\alpha_n) = 1 + \sum \alpha_1 + \sum \alpha_1\alpha_2 + \sum \alpha_1\alpha_2\alpha_3 + \cdots + \alpha_1\alpha_2\alpha_3 \dots \alpha_n$$

$$= 1 + p_1 + p_2 + p_3 + \cdots + p_n.$$

Problem 2.

If α, β, γ are the roots of the equation $x^3 + ax - b = 0$, find the value of

(i) $\sum \left(\frac{\alpha}{\beta\gamma}\right)$ (ii) $\sum \left(\frac{\alpha\beta}{\gamma}\right)$ (iii) $\sum \left(\frac{\alpha}{\beta+\gamma}\right)$

(iv) $\sum \left(\frac{1}{\beta+\gamma}\right)$ (v) $\sum \left(\frac{\beta}{\gamma} + \frac{\gamma}{\beta}\right)$ (vi) $\sum \alpha^3$

Solution:

We have $\sum \alpha = 0$ …………… (1)

$\sum \alpha\beta = a$…………. (2)

$\alpha\beta\gamma = b$…………… (3)

(i) $\sum\left(\frac{\alpha}{\beta\gamma}\right)=\frac{\alpha}{\beta\gamma}+\frac{\beta}{\gamma\alpha}+\frac{\gamma}{\alpha\beta}$

$$=\frac{\alpha^2+\beta^2+\gamma^2}{\alpha\beta\gamma}$$

$$=\frac{(\sum\alpha)^2-2(\sum\alpha\beta)}{\alpha\beta\gamma}$$

$$=-\frac{-2a}{b}\ \text{(by using (1) and (2))}$$

(ii) $\sum\left(\frac{\alpha\beta}{\gamma}\right)=\frac{\alpha\beta}{\gamma}+\frac{\beta\gamma}{\alpha}+\frac{\gamma\alpha}{\beta}$

$$=\frac{(\alpha\beta)^2+(\beta\gamma)^2+(\gamma\alpha)^2}{\alpha\beta\gamma}$$

$$=\frac{\sum\alpha^2\beta^2}{\alpha\beta\gamma}$$

$$=\frac{(\sum\alpha\beta)^2-2\alpha\beta\gamma(\sum\alpha)}{\alpha\beta\gamma}$$

$$=\frac{a^2}{b}\ \text{(by using (1), (2) and (3))}$$

(iii) $\sum\left(\frac{\alpha}{\beta+\gamma}\right)=\frac{\alpha}{\beta+\gamma}+\frac{\beta}{\gamma+\alpha}+\frac{\gamma}{\alpha+\beta}$

$$=\frac{\alpha}{\beta+\gamma}+\frac{\beta}{\gamma+\alpha}+\frac{\gamma}{\alpha+\beta}$$

$$=\frac{\alpha}{-\alpha}+\frac{\beta}{-\beta}+\frac{\gamma}{-\gamma}\ \text{(by using (1))}$$

$$=-3$$

(iv) $\sum\left(\frac{1}{\beta+\gamma}\right)=\frac{1}{\beta+\gamma}+\frac{1}{\gamma+\alpha}+\frac{1}{\alpha+\beta}$

$$=\frac{1}{-\alpha}+\frac{1}{-\beta}+\frac{1}{-\gamma}\ \text{(by using (1))}$$

$$=-\left(\frac{\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma}\right)$$

$$=-\frac{a}{b}\ \text{(by using (2) and (3))}$$

(v) $\sum\left(\frac{\beta}{\gamma}+\frac{\gamma}{\beta}\right)=\sum\left(\frac{\beta^2+\gamma^2}{\beta\gamma}\right)$

$$=\frac{\beta^2+\gamma^2}{\beta\gamma}+\frac{\gamma^2+\alpha^2}{\gamma\alpha}+\frac{\alpha^2+\beta^2}{\alpha\beta}$$

$$=\frac{\alpha(\beta^2+\gamma^2)+\alpha(\gamma^2+\alpha^2)+\alpha(\alpha^2+\beta^2)}{\alpha\beta\gamma}$$

$$=\frac{\alpha^2\beta+\alpha^2\gamma+\beta^2\gamma+\beta^2\alpha+\gamma^2\alpha+\gamma^2\beta}{\alpha\beta\gamma}$$

$$=\frac{\sum\alpha^2\beta}{\alpha\beta\gamma} \text{.................. (4)}$$

Now, multiplying equation (1) and (2), we get,

$$(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)=0$$

(ie) $\sum\alpha^2\beta+3\,\alpha\beta\gamma=0$

$$\therefore \quad \sum\alpha^2\beta=-3\,\alpha\beta\gamma$$

$\therefore$ From (4), we get,

$$\sum\left(\frac{\beta}{\gamma}+\frac{\gamma}{\beta}\right)=\frac{-3\,\alpha\beta\gamma}{\alpha\beta\gamma}=-3$$

(vi) $\sum\alpha^3=\alpha^3+\beta^3+\gamma^3$

$$=(\textstyle\sum\alpha)^3-3(\sum\alpha\beta)(\sum\alpha)$$

$$=0 \text{ (by using (1) and (3))}$$

Problem 3.

If $\alpha,\beta,\gamma,\delta$ are the roots of the equation $x^4+ax^2+bx+c=0$, find the value of $\sum\left(\frac{\beta+\gamma+\delta-\alpha}{2\alpha^2}\right)$.

Solution:

We have $\sum\alpha=0$ .... (1)

$\sum\alpha\beta=a$ .... (2)

$\sum\alpha\beta\gamma=-b$ .... (3)

$\alpha\beta\gamma\delta=c$ ....(4)

Now,

$$\sum\left(\frac{\beta+\gamma+\delta-\alpha}{2\alpha^2}\right)=\sum\left(\frac{\alpha+\beta+\gamma+\delta-2\alpha}{2\alpha^2}\right)$$

$$=\sum\left(\frac{0-2\alpha}{2\alpha^2}\right) \qquad \text{(by using (1))}$$

$$=-\sum\left(\frac{1}{\alpha}\right)$$

$$=-\left[\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}\right]$$

$$=-\left[\frac{\beta\gamma\delta+\alpha\gamma\delta+\alpha\beta\delta+\alpha\beta\gamma}{\alpha\beta\gamma\delta}\right]$$

$$=-\frac{\sum\alpha\beta\gamma}{\alpha\beta\gamma\delta}$$

$$=-\left(-\frac{b}{c}\right)$$

$$=\frac{b}{c} \qquad \text{(by using (3))}$$

Problem 4.

Show that the equation $x^3+qx+r=0$ will have one root twice another if

$343r^2+36q^3=0$.

Solution:

Let the roots be $\alpha, 2\alpha, \beta$.

$\therefore \quad S_1=\alpha+\ 2\alpha+\ \beta=0$

(ie)., $\qquad \beta=-3\alpha$ ………….. (1)

$S_2=2\alpha^2+\ 2\alpha\beta+\alpha\,\beta=q$

(ie)., $\qquad 2\alpha^2+\ 3\alpha\beta=q$ ………….. (2)

Using (1) in (2), we get, $-7\alpha^2=q$ ………….. (3)

Also, $S_3=2\alpha^2\beta=-r$

$\therefore \quad 6\alpha^3=r$ (by using (1))

$\therefore \quad \alpha^3 = \frac{r}{6}$ .... (4)

From (3), we get,

$$-7^3\alpha^6 = q^3 \quad \text{............. (5)}$$

Using (4) in (5), we get, $-343\left(\frac{r^2}{36}\right) = q^3$

(ie)., $\quad 343r^2 + 36q^3 = 0.$

EXERCISES:

1. If $\alpha, \beta, \gamma, \delta$ are the roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$, find the value of

(i) $\sum\left(\frac{1}{\alpha}\right)$ (ii) $\sum\left(\frac{\alpha}{\beta}\right)$ (iii) $\sum\left(\frac{1}{\alpha\beta}\right)$

(iv) $\sum \alpha^2$ (v) $\sum \alpha^2 \beta$ (vi) $\sum \alpha^2 \beta\gamma$

(vii) $\sum \alpha^2 \beta^2$ (viii) $\sum \alpha^3$ (ix) $\sum \alpha^4$

(vii) $\sum\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)$ (viii) $(\alpha+\beta+\gamma)(\beta+\gamma+\delta)(\gamma+\delta+\alpha)(\delta+\alpha+\beta)$.

2. If α, β, γ are the roots of the equation $x^3 + px^2 + qx + r = 0$, find the value of $(1+\alpha^2)(1+\beta^2)(1+\gamma^2)$.

3. Solve the equation $4x^3 - 24x^2 + 23x + 18 = 0$, given that the roots are in arithmetic progression.

4. If the sum of two roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$ equals the sum of the other two prove that $p^3 + 8r = 4pq$.

5. If the product of two roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$ is equal to the product of the other two show that $r^2 = p^2 s$.

6. Solve the equation $2x^3 + 3x^2 - 8x + 3 = 0$ given that the root is double the other one.

1.4. RECIPROCAL EQUATIONS:

In this section, we discuss the method of solving a class of equation called reciprocal equations.

Definition-1.4.1:

The equation $f(x) = 0$ is called **reciprocal equation (R.E)** if whenever α is a root of the equation, $\frac{1}{\alpha}$ is also a root.

Thus if $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ are the roots of a R. E, then the numbers $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \frac{1}{\alpha_3}, \dots, \frac{1}{\alpha_n}$ are the same as $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ in some order.

Examples:

(i)$x^2 + 2x + 1 = 0$ is a R. E and its roots are $-1, -1$.

(ii) $2x^2 - 5x + 2 = 0$ is a R. E and its roots are $2, \frac{1}{2}$.

(iii) $2x^3 + 3x^2 - 3x - 2 = 0$ is a R. E and its roots are $-2, -\frac{1}{2}, 1$.

Theorem-1.4.2:

If $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ are the roots of $f(x) = 0$, then $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \frac{1}{\alpha_3}, \dots, \frac{1}{\alpha_n}$ are the roots of $x^n f\left(\frac{1}{x}\right) = 0$.

Proof:

Since $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ are the roots of $f(x) = 0$, we have $f(\alpha_i) = 0$.

Let $g(x) = x^n f\left(\frac{1}{x}\right)$.

Then, $g\left(\frac{1}{\alpha_i}\right) = \left(\frac{1}{\alpha_i}\right)^n f(\alpha_i) = 0$ and hence $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \frac{1}{\alpha_3}, \dots, \frac{1}{\alpha_n}$ are the roots of $g(x)$.

Hence the result.

Remark:

If $f(x) = 0$ is a R. E then $f(x) = 0$ and $x^n f\left(\frac{1}{x}\right) = 0$ both has the same roots.

Theorem-1.4.3:

$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$ is a R.E if and only if $a_{n-r} = \pm\, a_r \quad (0 \le r \le n)$.

Proof:

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0.$$

Hence $x^n f\left(\frac{1}{x}\right) = x^n \left[a_0\left(\frac{1}{x}\right)^n + a_1\left(\frac{1}{x}\right)^{n-1} + \cdots + a_n\right]$

$$= a_nx^n + a_{n-1}x^{n-1} + \cdots + \ a_0.$$

Now,

Suppose $f(x) = 0$ is a R. E. Then $f(x) = 0$ and $x^n f\left(\frac{1}{x}\right) = 0$ both have the same roots and hence the corresponding coefficients of the two equations are proportional.

$$\therefore \quad \frac{a_0}{a_n} = \frac{a_1}{a_{n-1}} = \cdots = \frac{a_r}{a_{n-r}} = \cdots = \frac{a_n}{a_0} = k.$$

Since $\frac{a_0}{a_n} = \frac{a_n}{a_0} = k$, we get $k^2 = 1$ so that $k = \pm 1$.

$$\therefore \quad \frac{a_r}{a_{n-r}} = \pm 1 \text{ and hence } a_r = \pm\, a_{n-r}.$$

Conversely, suppose that $a_r = \pm\, a_{n-r}$.

Then the equation $f(x) = 0$ and $x^n f\left(\frac{1}{x}\right) = 0$ are same and hence $f(x) = 0$ is a R. E.

Definition-1.4.4:

A R. E $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$ is said to be of first type if $a_{n-r} = a_r$ and is said to be of second type if $a_{n-r} = -\, a_r$.

Remark:

$f(x) = 0$ is a R. E of first type iff $f(x) = x^n f\left(\frac{1}{x}\right)$ and is a R. E of second type iff $f(x) = -x^n f\left(\frac{1}{x}\right)$.

Definition-1.4.5:

A R. E $f(x) = 0$ is called a **standard reciprocal equation (S. R. E)** if it is of first type and the degree of $f(x)$ is even.

Examples:

(i) $4x^4 - 20x^3 + 33x^2 - 20x + 4 = 0$ is a S. R. E.

(ii) $6x^5 + 11x^4 - 33x^3 - 33x^2 + 11x + 6 = 0$ is a R. E of first type and odd degree.

(iii) $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0$ is a R. E of second type and odd degree.

SOLVED PROBLEMS:

Problem 1.

Show that $4(x^2 - x + 1)^3 = 27x^2(x - 1)^2$ is a standard reciprocal equation.

Solution:

Let $f(x) = 4(x^2 - x + 1)^3 - 27x^2(x - 1)^2$.

Now, $f\left(\frac{1}{x}\right) = 4\left(\left(\frac{1}{x}\right)^2 - \frac{1}{x} + 1\right)^3 - 27\left(\frac{1}{x}\right)^2\left(\frac{1}{x} - 1\right)^2$

$= \frac{4(1-x+x^2)^3}{x^6} - \frac{27(1-x)^2}{x^4}$

$\therefore \quad x^6 f\left(\frac{1}{x}\right) = 4(1 - x + x^2)^3 - 27x^2(1 - x)^2$

$= f(x).$

Also, $f(x)$ is of degree 6.

$\therefore \quad f(x)$ is a S. R. E.

Problem 2.

Solve $4x^4 - 20x^3 + 33x^2 - 20x + 4 = 0.$

Solution:

Given $4x^4 - 20x^3 + 33x^2 - 20x + 4 = 0$ ……………..(1)

The given equation is a S. R. E.

Dividing the equation by x^2 and regrouping, we get,

$$4\left(x^2 + \frac{1}{x^2}\right) - 20\left(x + \frac{1}{x}\right) + 33 = 0 \quad \text{……………..(2)}$$

Put $x + \frac{1}{x} = y$

Hence,

$$x^2 + \frac{1}{x^2} = y^2 - 2.$$

$\therefore$ (2) becomes $4(y^2 - 2) - 20y + 33 = 0$

(ie)., $4y^2 - 20y + 25 = 0$

(ie)., $(2y - 5)^2 = 0$

$\therefore\ y = \frac{5}{2}, \frac{5}{2}.$

Now,

$x + \frac{1}{x} = \frac{5}{2} \quad \Rightarrow \ 2x^2 - 5x + 2 = 0$

$\Rightarrow \ (2x - 1)(x - 2) = 0$

$\Rightarrow \ x = \frac{1}{2}, 2$

$\therefore$ The roots of (1) are $2, \frac{1}{2}, 2, \frac{1}{2}.$

Problem 3.

Solve $6x^5 + 11x^4 - 33x^3 - 33x^2 + 11x + 6 = 0.$

Solution:

Let $f(x) = 6x^5 + 11x^4 - 33x^3 - 33x^2 + 11x + 6 = 0$. ……………..(1)

This is a R. E of first type and of odd degree.

Hence $x + 1$ is a factor of $f(x)$.

By actual division, we get,

$$f(x) = (x + 1)(6x^4 + 5x^3 - 38x^2 + 5x + 6) = 0.$$

Now,

$(6x^4 + 5x^3 - 38x^2 + 5x + 6) = 0$ is a S. R. E.

Dividing by x^2 and regrouping, we get,

$$6\left(x^2 + \frac{1}{x^2}\right) + 5\left(x + \frac{1}{x}\right) - 38 = 0 \quad \text{……………..(2)}$$

Put $x + \frac{1}{x} = y$

Hence,

$$x^2 + \frac{1}{x^2} = y^2 - 2.$$

$\therefore$ (2) becomes $6(y^2 - 2) + 5y - 38 = 0$

(ie)., $6y^2 + 5y - 50 = 0$

(ie)., $(3y + 10)(2y - 5) = 0$

$\therefore\ y = -\frac{10}{3}$ and $y = \frac{5}{2}$.

Now,

Taking $x + \frac{1}{x} = -\frac{10}{3}$

$\Rightarrow\ 3x^2 + 10x + 3 = 0$

$\Rightarrow\ (3x + 1)(x + 3) = 0$

$\Rightarrow\ x = -3, -\frac{1}{3}$

Taking $x + \frac{1}{x} = \frac{5}{2}$ $\Rightarrow$ $2x^2 - 5x + 2 = 0$

$\Rightarrow$ $(2x - 1)(x - 2) = 0$

$\Rightarrow$ $x = \frac{1}{2}, 2$

$\therefore$ The roots of $f(x) = 0$ are $-1, -\frac{1}{3}, -3, 2, \frac{1}{2}$.

Problem 4.

Solve $2x^5 - 15x^4 + 37x^3 - 37x^2 + 15x - 2 = 0$.

Solution:

Let $f(x) = 2x^5 - 15x^4 + 37x^3 - 37x^2 + 15x - 2 = 0$. ……………..(1)

This is a R. E of second type and of odd degree.

Hence $x - 1$ is a factor of $f(x)$.

By actual division, we get,

$$f(x) = (x - 1)(2x^4 - 13x^3 + 24x^2 - 13x + 2) = 0.$$

Now,

$(2x^4 - 13x^3 + 24x^2 - 13x + 2) = 0$ is a S. R. E.

Dividing by x^2 and regrouping, we get,

$$2\left(x^2 + \frac{1}{x^2}\right) - 13\left(x + \frac{1}{x}\right) + 24 = 0 \qquad \text{……………..(2)}$$

Put $x + \frac{1}{x} = y$

Hence,

$$x^2 + \frac{1}{x^2} = y^2 - 2.$$

$\therefore$ (2) becomes $2(y^2 - 2) - 13y + 24 = 0$

(ie)., $2y^2 - 13y - 20 = 0$

(ie)., $(2y-5)(y-4)=0$

$\therefore\ y=4$ and $y=\frac{5}{2}$.

Now,

Taking $x+\frac{1}{x}=4 \quad \Rightarrow\ x^2-4x+1=0$

$\Rightarrow\ x=\frac{4\pm\sqrt{16-4}}{2}$

$\Rightarrow\ x=\frac{4\pm\sqrt{12}}{2}$

$\Rightarrow\ x=2\pm\sqrt{3}$

$\Rightarrow\ x=2+\sqrt{3}$ and $x=2-\sqrt{3}$

Taking $x+\frac{1}{x}=\frac{5}{2} \quad \Rightarrow\ 2x^2-5x+2=0$

$\Rightarrow\ (2x-1)(x-2)=0$

$\Rightarrow\ x=\frac{1}{2},2$

$\therefore$ The roots of $f(x)=0$ are $2+\sqrt{3}, 2-\sqrt{3}, 2, \frac{1}{2}$.

EXERCISES:

1. Solve $3x^6+x^5-27x^4+27x^2-x-3=0$.

2. Solve $2x^7-x^6-3x^4-3x^3-x+2=0$.

3. Solve $x^4-10x^3+26x^2-10x+1=0$.

4. Solve $6x^5-x^4-43x^3+43x^2+x-6=0$.

5. Solve $x^5-6x^4+13x^3-13x^2+6x-1=0$.

6. Solve $x^5-6x^4+7x^3+7x^2-6x+1=0$.

Unit-II:

Transformation of Equations – Approximate solutions to equations – Newton's method and Horner's method.

2.1 TRANSFORMATION OF EQUATIONS

Introduction:

If $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ are the roots of $f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$, $a_n \neq 0$. We form a new equation whose roots are some functions of roots of $f(x) = 0$. This formation of new equation is known as transformation of equations.

2.1.1 Formation of the equation whose roots are k times the roots of $f(x) = 0, (k \neq 0)$:

If $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ are the roots of

$f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$ then the equation whose roots are $k\alpha_1, k\alpha_2, k\alpha_3, \dots, k\alpha_n$ where $k \neq 0$ is given by $f\left(\frac{x}{k}\right) = 0$. …………………(1)

(ie)., $$\left(\frac{x}{k}\right)^n + a_1\left(\frac{x}{k}\right)^{n-1} + a_2\left(\frac{x}{k}\right)^{n-2} + \cdots + a_{n-1}\left(\frac{x}{k}\right) + a_n = 0$$

$$\therefore \left(\frac{1}{k^n}\right)[x^n + a_1kx^{n-1} + a_2k^2x^{n-2} + \cdots + a_{n-1}k^{n-1}x + k^na_n] = 0$$

$$x^n + a_1kx^{n-1} + a_2k^2x^{n-2} + \cdots + a_{n-1}k^{n-1}x + k^na_n = 0.$$

Hence the new equation is obtained by multiplying the successivecoefficients of the given equation (from the left) by $1, k, k^2, \dots, k^n$ respectively.

Note:

Putting $k = -1$ in (1) we get the equation whose roots are $-\alpha_1, -\alpha_2, -\alpha_3, \dots, -\alpha_n$ as

$$f(-x) = (-x)^n + a_1(-x)^{n-1} + a_2(-x)^{n-2} + \cdots + a_{n-1}(-x) + a_n = 0.$$

SOLVED PROBLEMS:

Problem 1.

Transform the equation $x^3 + 3x^2 + x - 4 = 0$ into the equation whose roots are multiplied by 10.

Solution:

The required equation is got by multiplying the successive coefficients $1,3,1,-4$ by $1, 10, 10^2, 10^3$ respectively.

Hence the required equation is $x^3 + 30x^2 + 100x - 4000 = 0$.

Problem 2.

Form the equation whose roots are negative of the roots of $x^3 - x^2 + x - 4 = 0$.

Solution:

The required equation is $f(-x) = 0$.

$\therefore$ It is given by $(-x)^3 - (-x)^2 + (-x) - 4 = 0$.

(ie)., $x^3 + x^2 + x + 4 = 0$.

2.1.2 Formation of the equation whose roots are diminished by h:

If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are the roots of $f(x) = 0$ then the equation whose roots are $\alpha_1 - h, \alpha_2 - h, \alpha_3 - h, \ldots, \alpha_n - h$ is given by $f(x + h) = 0$. …………………(1)

(ie)., $(x + h)^n + a_1(x + h)^{n-1} + a_2(x + h)^{n-2} + \cdots + a_{n-1}(x + h) + a_n = 0$

$\therefore$ $x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_{n-1}x + b_n = 0$ where $b_1, b_2, b_3, \ldots, b_n$ are constants to be determined. The coefficients $b_n, b_{n-1}, b_{n-2}, \ldots, b_1$ (in this order) are the reminders obtained by dividing $f(x)$ by $x - h$ in succession n times.

Note:

To increase the roots of $f(x)$ by h we diminish the roots of $f(x)$ by $-h$.

SOLVED PROBLEMS:

Problem 1.

Diminish the roots of the equation $x^3 + x^2 + x - 100 = 0$ by 4.

Solution:

4	1	1	1	-100
		4	20	84
	1	5	21	$\mathbf{-16}$
		4	36	
	1	9	**57**	
		4		
	1	**13**		

$\therefore$ The transformed equation is $x^3 + 13x^2 + 57x - 16 = 0$

Problem 2.

Increase the roots of the equation $4x^5 + 13x^2 + 57x - 10 = 0$ by 2.

Solution:

Here we diminish the roots by -2.

-2	4	0	-2	0	7	-3
		-8	16	-28	56	-126
	4	-8	-14	-28	63	**-129**
		-8	32	-92	240	
	4	-16	46	-120	**303**	
		-8	48	-188		
	4	-24	94	**-308**		
		-8	64			
	4	-32	**158**			
		-8				
	4	**-40**				

$\therefore$ The transformed equation is $4x^5 - 40x^4 + 158x^3 - 308x^2 + 303x - 129 = 0$

EXERCISES:

1. Multiply the roots of the equation $3x^3 - 2x^2 - x + 1 = 0$ by 4.

2. Multiply the roots of the equation $3x^3 - 10x^2 + 9x + 2 = 0$ by 3.

3. Multiply the roots of the equation $x^4 + 2x^3 + 4x^2 + 6x + 8 = 0$ by $\frac{1}{2}$.

4. Find the equation whose roots are equal in magnitude but opposite in sign to the roots of the equations.

(i) $x^5 - 3x^3 + 2x^2 + 5x + 5 = 0$.

(ii) $x^{10} + 12x^8 + 40x^4 - 15x + 20 = 0$.

5. Diminish the roots of $x^4 + 3x^3 - 2x^2 - 4x - 3 = 0$ by 3.

6. Diminish the roots of $x^4 - 5x^3 + 7x^2 - 4x + 5 = 0$ by 2.

2.2 APPROXIMATE SOLUTIONS TO EQUATIONS

In this section we describe two methods, due to Newton and Horner, for finding approximate values of the irrational roots of equation of the form $f(x) = 0$ where $f(x)$ is a polynomial. Throughout this section we are only concerned with positive roots.

To find the negative roots of $f(x) = 0$, we find the positive roots of $f(-x) = 0$.

Note:

Suppose a single unrepeated root of $f(x) = 0$ lies in the interval $(a, a + 1)$ then a can be located by using the condition that $f(a)$ and $f(a + 1)$ are of opposite signs.

2.3 NEWTON'S METHOD:

Suppose that the equation $f(x) = 0$ has a single unrepeated root in an interval (α, β) where $\beta - \alpha$ is small. Hence $f'(x) \neq 0$ for all $x \in (\alpha, \beta)$.

Now, let $\alpha_1 = \alpha + h$ be the exact value of the root.

$f(\alpha + h) = 0.$

Hence by Taylor's expansion we have

$$f(\alpha)+\frac{h}{1!}f'(\alpha)+\frac{h^2}{2!}f''(\alpha)+\cdots=0.$$

Omitting h^2 and higher power of h we get $f(\alpha)+hf'(\alpha)=0$.

$$\therefore \quad h=-\frac{f(\alpha)}{f'(\alpha)}$$

$\therefore \quad \alpha_1=\alpha-\frac{f(\alpha)}{f'(\alpha)}$ is an approximate value of the root.

By repeating this process, the approximation can be carried out to any required degree of accuracy.

SOLVED PROBLEMS:

Problem 1.

Show that $x^3+3x-1=0$ has only one real root and calculate it correct to two places of decimals.

Solution:

Since there is only one change of sign there cannot be more than one positive root.

Further changing x to $-x$ the equation becomes $-x^3-3x-1=0$.

(ie)., $x^3+3x+1=0$.

This equation has no change of sign. Hence it has no negative root.

Since the imaginary roots occur in conjugate pairs the given equation has two imaginary roots and only one real root.

Now,

We find the real root by Newton's method approximately to two places of decimals.

Let $f(x)=x^3+3x-1$.

Since $f(0)=-1<0$ and $f(1)=3>0$, the root lies between 0 and 1 (first approximation).

Let $0 + h$ be the actual value of the root.

$$\therefore \quad h \approx -\frac{f(0)}{f'(0)}.$$

Now,

$$f'(x) = 3x^2 + 3.$$

$$\therefore \quad f'(0) = 3.$$

$$\therefore \quad h \approx \frac{1}{3} \approx 0.3.$$

Hence the root is $0 + 0.3 \approx 0.3$ (second approximation).

For third approximation, let the root be

$$0 + h_1 \text{ where } h_1 \approx -\frac{f(0.3)}{f'(0.3)}.$$

$$h_1 \approx \frac{(0.3)^3+3(0.3)-1}{3(0.3)^2+3}.$$

$$\approx 0.022.$$

$$\therefore \quad \text{The root is} \approx 0.3 + 0.022.$$

$$\approx 0.32 \text{ (up to two places of decimals).}$$

Problem 2.

Find correct to 2 places of decimals the root of the equation $x^4 - 3x + 1 = 0$ that lies between 1 and 2.

Solution:

We use Newton's method to find the approximate value of the root.

Let $f(x) = x^4 - 3x + 1$

Since the required root lies between 1 and 2.

Let $1 + h$ be the actual value of the root where $h \approx -\frac{f(1)}{f'(1)}$.

Now,

$$f(1) = -1.$$

Also,

$$f'(x) = 4x^3 - 3.$$

$$\therefore \quad f'(1) = 1.$$

$$\therefore \quad h \approx -\left(\frac{-1}{1}\right) = 1.$$

$\therefore$ h is not small compared to the integral part of the root 1.

$\therefore$ We find the value of root still closer to 1.

We observe that $f(1.3) < 0$ and $f(1.4) > 0$. (verify)

Hence the root is lies between 1.3 and 1.4.

Let $\alpha_1 = 1.3 + h_1$ be the actual value of the root.

$$\therefore \quad h_1 \approx -\frac{f(1.3)}{f'(1.3)}$$

$$= -\left(\frac{-0.0439}{5.788}\right).$$

$$\approx 0.008.$$

$$\therefore \quad \alpha_1 + h_1 \approx 1.3 + 0.008$$

$$\approx 1.31. \text{ (up to two places of decimals)}.$$

$\therefore$ The root is ≈ 1.31. (up to two places of decimals).

Problem 3.

Obtain by Newton's method the root of the equation $x^3 - 3x + 1 = 0$ that lies between 1 and 2.

Solution:

Let $f(x) = x^4 - 3x + 1$

$\therefore\ f'(x) = 3x^2 - 3.$

Since the required root lies between 1 and 2,

let it be $\alpha + h = 1 + h$, where $h \approx -\frac{f(\alpha)}{f'(\alpha)}$.

$$\therefore \quad h \approx -\frac{f(1)}{f'(1)}.$$

Since $f'(1) = 0$, we take another closer value of $\alpha = 1$ than 1.

We observe that $f(1.5) = -0.125 < 0$ and $f(2) = 3 > 0$. (verify)

Hence the root is lies between 1.5 and 2.

Let $1.5 + h_1$ be the actual value of the root.

$$\therefore \quad h_1 \approx -\frac{f(1.5)}{f'(1.5)}$$

$$= -\left(\frac{-0.125}{3.75}\right).$$

$$\approx 0.033.$$

$$\therefore \quad \alpha + h_1 \approx 1.5 + 0.033$$

$$\approx 1.533.$$

Up to next order approximation the root is

$$= 1.533 + \left[-\frac{f(1.533)}{f'(1.533)}\right]$$

$$\approx 1.5327. \quad \text{(verify)}$$

$$\approx 1.53. \quad \text{(up to two places of decimals)}.$$

$\therefore$ The root is ≈ 1.53. (up to two places of decimals).

Problem 4.

Find the negative root of the equation $x^3 - 2x + 5 = 0$ correct to two places of decimals.

Solution:

Let $g(x) = x^3 - 2x + 5$

We find the positive root of $g(-x) = 0$.

(ie)., $g(-x) = -x^3 + 2x + 5 = 0$.

(ie)., $x^3 - 2x - 5 = 0$.

Let $f(x) = x^3 - 2x - 5$.

$\therefore f'(x) = 3x^2 - 2$.

We observe that $f(2) = -1 < 0$ and $f(3) = 16 > 0$. (verify)

Hence the root is lies between 2 and 3. (first approximation).

Let the actual value of the root be $2 + h$,

$$\text{where } h \approx -\frac{f(2)}{f'(2)}.$$

$$= \frac{2^3-4-5}{32^2-2}$$

$$= \frac{1}{10}$$

$$= 0.1.$$

$\therefore$ The root is 2.1 (second approximation).

For third approximation, let the root be

$$2.1 + h_1 \text{ where } h_1 \approx -\frac{f(2.1)}{f'(2.1)}.$$

$$h_1 \approx -\frac{9.261-4.2-5}{3(2.1)^2}.$$

$$\approx -\frac{0.061}{11.23}$$

$\approx -0.0054.$

$\therefore$ The positive root of $f(x) = 0$ is 2.09. (up to two places of decimals).

Hence, the negative root of $x^3 - 2x + 5 = 0$ is -2.09. (up to two places of decimals).

2.4 HORNER'S METHOD:

Horner's method is the most convenient way of finding approximate values of the irrational roots of the equation $f(x) = 0$ where $f(x)$ is the polynomial. The root is calculated in decimal form and the figures of the decimal are obtained in succession. We describe below the steps to be followed.

Step I:

Consider the equation $f(x) = 0$. Suppose this has a single root α is the interval $(a, a + 1)$ where a is a positive integer. Then a can be located by using the condition that $f(a)$ and $f(a + 1)$ are of opposite signs.

Step II:

Suppose the exact value of the root is $a.\,a_1a_2$ ... Diminish the roots of $f(x) = 0$ by a. Then we get the transformed equation $f(x) = 0$ having $0.\,a_1a_2$ ... as a root.

Step III:

Multiply the roots of $f_1(x) = 0$ by 10 and we obtain the transformed equation $f_2(x) = 0$ having $a_1.\,a_2a_3$ ... as a root.

Step IV:

By inspection we locate the root by finding two consecutive integers b and $b + 1$ such that $f_2(b)$ and $f_2(b + 1)$ are of opposite signs. Then $b = a_1$ is the first decimal in the root making $a.\,a_1$ as the first approximation of the root.

Repeat this process (steps I to IV) as many times as needed to get the roots of $f(x) = 0$ to any desired number of decimal places.

Unit-III:

Matrices – Characteristic equation of a matrix – Eigenvalues and Eigen vectors– Cayley Hamilton theorem and simple problems.

3.1 MATRICES

Introduction:

A matrix is a collection of numbers arranged in the form of a rectangular array. Such arrays occur in various branches of pure and applied Mathematics. For example, the coefficients of a system of simultaneous linear equations can be represented as a matrix. In this chapter we introduce the basic concepts for matrices and discuss problems such as consistency of a system of simultaneous linear equations and eigen value problems.

The reader is familiar with the basic concept of matrices, matrix addition, matrix multiplication and rank of a matrix. In this section we briefly recall the basic concepts.

A $matrix$ A is an array of mn numbers a_{ij} where $1 \leq i \leq m, 1 \leq j \leq n$ arranged in m rows n columns as follows.

$$\begin{pmatrix} a_{11} & a_{12} \ldots & a_{1n} \\ a_{21} & a_{22} \ldots & a_{23} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} \ldots & a_{mn} \end{pmatrix}$$

We shall denote this matrix by the symbol (a_{ij}). If $m = n$ then A is called a $square\ matrix$ of order n.

We define the addition of two $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ by

$A + B = (a_{ij} + b_{ij})$.

We note that we can add two matrices if and only if they have the same number of rows and columns.

A square matrix $A = (a_{ij})$ is said to be $symmetric$ if $a_{ij} = a_{ji}$ for all $i,\ j$.

A square matrix A is said to be $singular$ if the determinant of the matrix, $|A| = 0$. A is called a $non-singular$ matrix if $|A| \neq 0$.

Let $A = (a_{ij})$ be a square matrix and let (A_{ij}) denote the co-factor of a_{ij}. Then the transpose of the matrix (A_{ij}) is called the *adjoint* or *adjugate* of the matrix A and denoted by $adj\ A$. Thus the $(i,j)^{th}$ entry of $adj\ A$ is A_{ij}.

If A is a square matrix and $|A| \neq 0$, then the *inverse* of A is given by

$A^{-1} = \frac{adj\ A}{|A|}$.

Problem 1.

Compute the inverse of the matrix

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}.$$

Solution:

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{vmatrix} = -1 \quad \text{(verify)}$$

Since $|A| \neq 0$, A^{-1} exists and is given by $A^{-1} = \frac{adj\ A}{|A|}$.

Now,

$$\text{we find } adj\ A = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$$

where $A_{ij} \quad (i,j = 1,2,3)$ are the cofactors of a_{ij}.

$$A_{11} = \begin{vmatrix} 6 & -5 \\ -2 & 2 \end{vmatrix} = 2;$$

$$A_{12} = \begin{vmatrix} -15 & -5 \\ 5 & 2 \end{vmatrix} = 5;$$

$$A_{13} = \begin{vmatrix} -15 & 6 \\ 5 & -2 \end{vmatrix} = 0;$$

$$A_{21} = \begin{vmatrix} -1 & 1 \\ -2 & 2 \end{vmatrix} = 0;$$

$A_{22} = \begin{vmatrix} 2 & 1 \\ 5 & 2 \end{vmatrix} = -1;$

$A_{23} = \begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix} = -1;$

$A_{31} = \begin{vmatrix} -1 & 1 \\ 6 & -5 \end{vmatrix} = -1;$

$A_{32} = \begin{vmatrix} 2 & 1 \\ -15 & -5 \end{vmatrix} = -5;$

$A_{33} = \begin{vmatrix} 2 & -1 \\ -15 & 6 \end{vmatrix} = -3;$

Hence $adj\ A = \begin{pmatrix} 2 & 0 & -1 \\ 5 & -1 & -5 \\ 0 & -1 & -3 \end{pmatrix}$

$$A^{-1} = \frac{1}{-1}\begin{pmatrix} 2 & 0 & -1 \\ 5 & -1 & -5 \\ 0 & -1 & -3 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 1 \\ -5 & 1 & 5 \\ 0 & 1 & 3 \end{pmatrix}$$

EXERCISES:

1.Compute the inverse of the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

2.Compute the inverse of the matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{pmatrix}$.

3.Compute the inverse of the matrix $A = \begin{pmatrix} 2 & 2 & -3 \\ -3 & 2 & 2 \\ 2 & -3 & 2 \end{pmatrix}$.

3.2 CHARACTERISTIC EQUATION OF A MATRIX:

Definition- 3.2.1:

The equation $|A - \lambda I| = 0$, is called the *characteristic equation* of the matrix A.

Note:

1. Solving $|A - \lambda I| = 0$, we get n roots for λ and these roots are called *characteristic roots or eigen values or latent values* of the matrix A.

2. Corresponding to each value of λ, the equation $AX = \lambda X$ has a non-zero solution vector X.

If X , be the non-zero vector satisfying $AX = \lambda X$, when $\lambda = \lambda_r$, X_r is said to be the latent vector or eigen vector of a matrix A corresponding to λ_r.

3.2.2 CHARACTERISTIC POLYNOMIAL:

The determinant $|A - \lambda I|$ when expanded will give a polynomial, which we call as $characteristic\ polynomial$ of matrix A.

Working rule to find characteristic equation:

For a 3 x 3 matrix:

Method 1:

The characteristic equation is $|A - \lambda I| = 0$.

Method 2:

Its characteristic equation can be written as $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_1 = 0$,

where

S_1 = Sum of the main diagonal elements

S_2 = Sum of the minors of the main diagonal elements

S_3 = Determinants of $A = |A|$.

For a 2 x 2 matrix:

Method 1:

The characteristic equation is $|A - \lambda I| = 0$.

Method 2:

Its characteristic equation can be written as $\lambda^2 - S_1\lambda + S_2 = 0$,

where

S_1 = Sum of the main diagonal elements

S_2 = Determinants of $A = |A|$.

SOLVED PROBLEMS:

Problem 1.

Find the characteristic equation of the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

Solution:

Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$.

Its characteristic equation is $\lambda^2 - S_1\lambda + S_2 = 0$,

where

S_1 = Sum of the main diagonal elements

S_2 = Determinants of $A = |A|$.

Now,

$S_1 = 1 + 2 = 3$.

$S_2 = 1(2) - 2(0) = 2$.

Therefore,

the characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$.

Problem 2.

Find the characteristic equation of the matrix $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$

Solution:

Let $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$.

Its characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_1 = 0$,

where

S_1 = Sum of the main diagonal elements

S_2 = Sum of the minors of the main diagonal elements

S_3 = Determinants of $A = |A|$.

Now,

$$S_1 = 8 + 7 + 3 = 18.$$

$$S_2 = \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}.$$

$$S_3 = 8(5) + 6(-10) + 2(10)$$

$$= 40 - 60 + 20$$

$$= 0.$$

Therefore,

the characteristic equation is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$.

Problem 3.

Find the characteristic polynomial of the matrix $\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$

Solution:

Let $A = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$.

Its characteristic equation is $\lambda^2 - S_1\lambda + S_2 = 0$,

where

S_1 = Sum of the main diagonal elements

S_2 = Determinants of $A = |A|$.

Now,

$$S_1 = 3 + 2 = 5.$$

$$S_2 = 3(2) - 1(-1) = 7.$$

Therefore,

the characteristic polynomial is $\lambda^2 - 5\lambda + 7$.

3.3 CAYLEY HAMILTON THEOREM AND SIMPLE PROBLEMS:

Statement:

Every square matrix satisfies its own characteristic equation

Uses of Cayley-Hamilton theorem:

(1) To calculate the positive integral powers of A.

(2) To calculate the inverse of a square matrix A.

SOLVED PROBLEMS:

Problem 1.

Show that the matrix $\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$ satisfies its own characteristic equation.

Solution:

Let $A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$.

Its characteristic equation is $\lambda^2 - S_1\lambda + S_2 = 0$

where

$S_1 =$ Sum of the main diagonal elements

$S_2 =$ Determinants of $A = |A|$.

Now,

$S_1 = 1 + 1 = 2.$

$S_2 = 1 - (-4) = 5.$

Therefore,

the characteristic equation is $\lambda^2 - 2\lambda + 5 = 0$.

To prove: $A^2 - 2A + 5I = 0$.

$$A^2 = A(A) = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -4 \\ 4 & -3 \end{pmatrix}$$

$$\therefore \quad A^2 - 2A + 5I = \begin{pmatrix} -3 & -4 \\ 4 & -3 \end{pmatrix} - \begin{pmatrix} 2 & -4 \\ 4 & 2 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore,

the given matrix satisfies its own characteristic equation.

Problem 2.

If $A = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$ write A^2 in terms of A and I, using Cayley-Hamilton theorem.

Solution:

Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic equation.

The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$

where

S_1 = Sum of the main diagonal elements

S_2 = Determinants of $A = |A|$.

Now,

$S_1 = 6.$

$S_2 = 5.$

Therefore,

the characteristic equation is $\lambda^2 - 6\lambda + 5 = 0$.

By Cayley-Hamilton theorem, $A^2 - 6A + 5I = 0$.

$$\text{(ie).,} \ A^2 = 6A - 5I.$$

Problem 3.

Verify Cayley-Hamilton theorem, find A^4 and A^{-1} when $A = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$.

Solution:

$$\text{Let } A = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$

The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_1 = 0$,

where

S_1 = Sum of the main diagonal elements

S_2 = Sum of the minors of the main diagonal elements

S_3 = Determinants of $A = |A|$.

Now,

$S_1 = 2+2+2 = 6.$

$S_2 = 3+2+3.$

$= 8$

$S_3 = 2(3)-1-2$

$= 3.$

Therefore,

the characteristic equation is $\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0.$

To prove: $A^3 - 6A^2 + 8A - 3I = 0$ ....(1)

$$A^2 = A(A) = \begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix}$$

$$A^3 = A^2(A) = \begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix}\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{pmatrix}$$

Now,

$A^3 - 6A^2 + 8A - 3I$

$$= \begin{pmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{pmatrix} - \begin{pmatrix} 42 & 36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{pmatrix} + \begin{pmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{pmatrix}$$

$$- \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$= 0$

To find A^3:

$$(1) \Rightarrow A^3 - 6A^2 + 8A - 3I = 0$$

$$\Rightarrow A^3 = 6A^2 - 8A + 3I$$

Therefore,

$$A^4 = 36\,A^2 - 48\,A + 18\,I - 8\,A^2 + 3\,A$$

$$= 28\,A^2 - 45\,A + 18\,I$$

Hence,

$$A^4 = 28\begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix} - 45\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} + 18\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 196 & -168 & 252 \\ -140 & 168 & -168 \\ 140 & -140 & 196 \end{pmatrix} - \begin{pmatrix} 90 & -45 & 90 \\ -45 & 90 & 45 \\ 45 & -45 & 90 \end{pmatrix} + \begin{pmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{pmatrix}$$

$$= \begin{pmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{pmatrix}$$

To find A^{-1}:

Multiplying (1) by A^{-1}, $A^2 - 6A + 8I - 3A^{-1} = 0$

$$\Rightarrow 3A^{-1} = A^2 - 6A + 8I$$

$$\Rightarrow 3A^{-1} = \begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix} - 6\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} + 8\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{pmatrix} + \begin{pmatrix} 12 & -6 & -12 \\ 6 & -12 & 6 \\ -6 & 6 & -12 \end{pmatrix} + \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{pmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{3}\begin{pmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{pmatrix}$$

EXERCISES:

1. Verify that $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ satisfies its own characteristic equation and hence find A^4.

2. Find A^{-1} if $A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$, using Cayley-Hamilton theorem.

3. If $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$, find A^n in terms of A.

3.4 EIGEN VALUES AND EIGEN VECTORS:

3.4.1 Eigen values and Eigen vectors of a real matrix:

Working rule to find eigen values and eigen vectors:

1. Find the characteristic equation $|A - \lambda I| = 0$.

2. Solve the characteristic equation to get characteristic roots. They are called $eigen\ values$.

3. To find the eigen vectors, solve $[A - \lambda I]X = 0$ for different values of λ.

Note:

1. Corresponding to n distinct eigen values, we get n independent eigen vectors

2. If 2 or more eigen values are equal, it may or may not be possible to get linearly independent eigen vectors corresponding to the repeated eigen values.

3. If X_i is a solution for an eigen value λ_i, then cX_i is also a solution, where c is an arbitrary constant. Thus, the eigen vector corresponding to an eigen value is not unique but may be any one of the vectors cX_i.

4. Algebraic multiplicity of an eigen value λ is the order of the eigen value as a root of the characteristic polynomial (i.e., if λ is a double root, then algebraic multiplicity is 2)

5. Geometric multiplicity of λ is the number of linearly independent eigen vectors corresponding to λ.

3.4.2 Non-symmetric matrix:

If a square matrix A is non-symmetric, then $A \neq A^T$.

Note:

1. In a non-symmetric matrix, if the eigen values are non-repeated then we get a linearly independent set of eigen vectors

2. In a non-symmetric matrix, if the eigen values are repeated, then it may or may not be possible to get linearly independent eigen vectors.

If we form a linearly independent set of eigen vectors, then diagonalization is possible through similarity transformation

3.4.2 Symmetric matrix:

If a square matrix A is symmetric, then $A = A^T$.

Note:

1. In a symmetric matrix, if the eigen values are non-repeated, then we get a linearly independent and pair wise orthogonal set of eigen vectors

2. In a symmetric matrix, if the eigen values are repeated, then it may or may not be possible to get linearly independent and pair wise orthogonal set of eigen vectors

If we form a linearly independent and pair wise orthogonal set of eigen vectors, then diagonalization is possible through orthogonal transformation

Unit-IV:

Differential equation of first order but of higher degree – Equations solvable for p, x, y – Partial differential equations – formations – solutions – Standard form $Pp + Qq = R$.

4.1 DIFFERENTIAL EQUATION OF FIRST ORDER BUT OF HIGHER DEGREE

The general form of differential equation of first order and higher degree is

$$\left(\frac{dy}{dx}\right)^n + P_1\left(\frac{dy}{dx}\right)^{n-1} + P_2\left(\frac{dy}{dx}\right)^{n-2} + \cdots + P_{n-1}\frac{dy}{dx} + P_n = 0.$$

where each P_i is a function of x and y. If $\frac{dy}{dx} = p$, then the general form reduces to

$$p^n + P_1p^{n-1} + P_2p^{n-2} + \cdots + P_{n-1}p + P_n = 0.$$

Hence it also can be written as $F(x, y, p) = 0$. In this chapter we study following methods of solving differential equation of first order and higher degree.

4.1.1 Method of solving differential equation of the form $F(x, y, p) = 0$.

1. Differential equations which are solvable for p.
2. Differential equations which are solvable for x.
3. Differential equations which are solvable for y.
4. Clairaut's differential equations.
5. Lagrange's differential equations.

4.2 DIFFERENTIAL EQUATIONS WHICH ARE SOLVABLE FOR p:

Suppose we can write the differential equation $F(x, y, p) = 0$ of degree n in the form

$$\big(p - f_1(x,y)\big)\big(p - f_2(x,y)\big)\big(p - f_3(x,y)\big) \ldots \big(p - f_n(x,y)\big) = 0 \text{ ....(1)}$$

Now,

comparing each factor with zero we get $\big(p - f_i(x,y)\big) = 0$, where $i = 1, 2, \ldots, n.$, which is linear differential equation. Suppose solution of $\big(p - f_i(x,y)\big) = 0$ is given by $F_i(x, y, c_i)) = 0$, where c_i is an arbitrary constant. Instead of taking different c_i's in the general solution of $\left(p - f_i(x,y)\right) = 0$ if we take only one c in all, then it makes no difference in general solution. Therefore, general solution $\left(p - f_i(x,y)\right) = 0$ will be $F_i(x,y,c)) = 0$. Then general solution of equation (1) is given by $F_1(x,y,c)F_2(x,y,c) \dots F_n(x,y,c) = 0$. Thus, differential equation of n degree and first order having linear factor $p - f_i(x,y) = 0$ are known as *solvable for p.*

SOLVED PROBLEMS:

Problem 1.

Solve: $xyp^3 + (x^2 - 2y^2)p^2 - 2xyp = 0$.

Solution:

The given differential equation is of degree 3 and therefore it has three linear factors.

$$p[xyp^2 + (x^2 - 2y^2)p - 2xy] = 0.$$

$$p[xyp^2 + x^2p - 2y^2p - 2xy] = 0.$$

$$\therefore \quad p(xp - 2y)(yp + x) = 0.$$

Comparing these three linear factors with zero, we get,

1. $p = 0 \Rightarrow y - c = 0.$
2. $xp - 2y = 0 \Rightarrow \frac{dy}{y} = 2\frac{dx}{y}.$

   $\Rightarrow y = cx^2.$
3. $yp + x = 0 \Rightarrow ydy + xdx = 0.$

   $\Rightarrow x^2 + y^2 - 2c = 0.$

Therefore, the general solution is given by multiplying these three solutions of linear factors of given equation.

$\therefore \quad (y - c)(y - cx^2)(x^2 + y^2 - 2c) = 0$, which is a general solution.

Problem 2.

Solve: $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$.

Solution:

The given differential equation is $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$ ……………………(1)

Put $p = \frac{dy}{dx}$ in (1),

we get, $p - \frac{1}{p} = \frac{x}{y} - \frac{y}{x}$

$\therefore \quad p^2 + p\left(\frac{y}{x} - \frac{x}{y}\right) - 1 = 0.$

$\therefore \quad \left(p + \frac{y}{x}\right)\left(p - \frac{x}{y}\right) = 0.$

Now,

Comparing these three linear factors with zero, we get,

1. $\frac{dy}{dx} + \frac{y}{x} = 0 \ \Rightarrow xdy + ydx = 0.$

   $\Rightarrow d(xy) = 0$

   $\Rightarrow xy = c.$

2. $\frac{dy}{dx} - \frac{x}{y} = 0 \ \Rightarrow xdy - ydy = 0.$

   $\Rightarrow x^2 - y^2 = c.$

Thus, the general solution can be obtained by multiplying the general solutions of the linear factors of given differential equation.

$\therefore \quad (xy - c)(x^2 - y^2 - c) = 0$, which is a general solution.

EXERCISES:

1. Solve $p^2 - (x + 3y)p + 2y(x + y) = 0.$

2. Solve $p^2 - 7p + 10 = 0.$

3. Solve $p(p+y) = x(x+y)$.

4. Solve $yp^2 + (x-y)p - x = 0$.

5. Solve $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$.

4.3 DIFFERENTIAL EQUATIONS WHICH ARE SOLVABLE FOR y:

If the differential equation of the form $F(x,y,p) = 0$ can be written as $y = f(x,p) = 0$, then it is said to be $solvable\ for\ y$. In order to solve these types of differential equation we differentiate with respect to x,
we get,

$$\frac{dy}{dx} = p = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}\frac{dp}{dx} = F\left(x, p, \frac{dp}{dx}\right) \qquad \text{....................(2)}$$

which is in variable p and x. Hence its solution is given by $g\ (x,p,c) = 0$. By eliminate p from equation (2) and $g\ (x,p,c)$ we get function $\phi(x,y,c)$ which will be the general solution of the given differential equation. If it is not possible to eliminate p, then general solution can be obtained by taking $x = F_1(p,c)$ and $y = F_2(p,c)$, where c is an arbitrary constant. Let us see following examples to understand this method.

SOLVED PROBLEMS:

Problem 1.

Solve: $xp^2 - 2yp + ax = 0$.

Solution:

The given differential equation is $xp^2 - 2yp + ax = 0$ ....(1)

Here,

$$y = \frac{1}{2}xp + \frac{1}{2}\frac{ax}{p}.$$

By differentiating with respect to x, we get,

$$\frac{dy}{dx} = \frac{1}{2}p + \frac{1}{2}x\frac{dp}{dx} + \frac{a}{2p} - \frac{ax}{2p^2}\frac{dp}{dx}.$$

$$\therefore\ p = \frac{1}{2}p + \left(\frac{1}{2}x - \frac{ax}{p^2}\right)\frac{dp}{dx} + \frac{1}{2}\ \frac{a}{p}.$$

$$\therefore\ p = \left(x - \frac{ax}{p^2}\right)\frac{dp}{dx} + \ \frac{a}{p}.$$

$$\Rightarrow p^3 - p^2 x\frac{dp}{dx} + ax\ \frac{dp}{dx} - ap = 0.$$

$$\therefore\ (p^3 - a)\left(p - x\frac{dp}{dx}\right) = 0.$$

$$p - x\frac{dp}{dx} = 0 \text{ or } p^3 - a = 0.$$

$$\therefore\ \frac{dp}{p} = \frac{dp}{x} \Rightarrow \log\ p = \log\ x + \log\ c.$$

$$\therefore\ p = cx.$$

Now, substitute $p = cx$ in $y = \frac{1}{2}xp + \frac{1}{2}\ \frac{ax}{p}$,

we get,

$y = \frac{1}{2}cx^2 + \frac{1}{2}\ \frac{a}{c}$, which is a general solution.

Problem 2.

Solve: $x + 2(xp - p) + p^2 = 0$.

Solution:

The given differential equation can be in the form $y = f(x, y)$.

Therefore, it is solvable for y.

Given $x + 2(xp - p) + p^2 = 0$. ……………………..(1)

(1) ⇒

$$y = \frac{1}{2}x + xp + \frac{1}{2}\ p^2.$$

By differentiating with respect to x, we get,

$$\frac{dy}{dx} = p = \frac{1}{2} + p + x\frac{dp}{dx} + p\ \frac{dp}{dx}.$$

$$\therefore\ (x + p)\frac{dp}{dx} + \frac{1}{2} = 0.$$

Now,

Put $x + p = u$,

we get,

$$1 + \frac{dp}{dx} = \frac{du}{dx}$$

$$\therefore\ u = \left(\frac{du}{dx} - 1\right) + \frac{1}{2} = 0.$$

$$\therefore\ \frac{du}{dx} = \frac{2u-1}{2u}.$$

$$\Rightarrow\ \frac{2u}{2u-1} du = dx.$$

$$\therefore\ \int\left(1 + \frac{1}{2u-1}\right) = \int dx + c.$$

$$u + \frac{1}{2}\log\,(2u - 1) = x + c.$$

$$x + p + +\frac{1}{2}\log\,(2x + 2p - 1) = x + c.$$

$$2p + +\frac{1}{2}\log\,(2x + 2p - 1) = c.$$

$$2x + 2p - 1 = e^{2p-c}.$$

$$x = \frac{1}{2}\, e^{2p-c} + 1 - p.$$

Here, we cannot eliminate p from above equation.

Hence,

the general solution can be obtained from $y = \frac{1}{2}x + xp + \frac{1}{2}\,p^2$ and $x = \frac{1}{2}\,e^{2p-c} + 1 - p$.

4.4 DIFFERENTIAL EQUATIONS WHICH ARE SOLVABLE FOR x:

If the differential equation of the form $F(x, y, p) = 0$ can be written as $x = f\,(y, p) = 0$, then it is said to be *solvable for x* . In order to solve these types of differential equation we differentiate with respect to y,

we get,

$$\frac{dy}{dx} = p = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial p}\,\frac{dp}{dy} = F\left(x, p, \frac{dp}{dy}\right) \qquad \text{....................}(3)$$

which is in variable p and y. Hence its solution is given by $g\ (y, p, c) = 0$. By eliminate p from equation (3) and $g\ (y, p, c)$, we get, function $\phi(x, y, c)$ which will be the general solution of the given differential equation. If it is not possible to eliminate p, then general solution can be obtained by taking $x = F_1(p, c)$ and $y = F_2(p, c)$, where c is an arbitrary constant. Let us see following examples to understand this method.

SOLVED PROBLEMS:

Problem 1.

Solve: $y^2p^2 - 3xp + y = 0$.

Solution:

Given $y^2p^2 - 3xp + y = 0$.

The given differential equation is of the form $x = f\ (y, p)$, where $f\ (y, p) = \frac{1}{3}\left(\frac{y}{p} + y^2p\right)$.

By differentiating with respect to y, we get,

$$3\frac{dx}{dy} = 3\frac{1}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} + 2yp + y^2\ \frac{dp}{dy}.$$

$$\therefore\ \ 2yp - \frac{2}{p} + \left(y^2 - \frac{y}{p^2}\right)\frac{dp}{dy} = 0.$$

$$2p(yp^2 - 1) + y(yp^2 - 1)\frac{dp}{dy} = 0.$$

$$(yp^2 - 1)\left(2p + y\frac{dp}{dy}\right) = 0.$$

We ignore $yp^2 - 1 = 0$, we get and consider $2p + y\frac{dp}{dy} = 0$.

$$\therefore\ \ \frac{dp}{p} + 2\frac{dy}{y} = 0.$$

Then,

$$\log\ p + 2\log\ y = \log\ c.$$

$$\therefore\ \ py^2 = c$$

$$\Rightarrow p = \frac{c}{y^2}.$$

Hence,

substitute value of p, we get $y^3 - 2cx + c^2 = 0$, which is a general solution.

Problem 2.

Solve: $x = p + \frac{1}{p}$.

Solution:

Given $x = p + \frac{1}{p}$.

It is easy to see that this differential equation is solvable for x.

By differentiating with respect to y,

we get,

$$\frac{dx}{dy} = \frac{1}{p} = \frac{dp}{dy} - \frac{1}{p^2}\frac{dp}{dy}.$$

$$\therefore \quad \frac{1}{p} = \left(1 - \frac{1}{p^2}\right)\frac{dp}{dy}.$$

$$\Rightarrow \left(\frac{p^2-1}{p}\right) dp = dy.$$

$$\therefore \int \left(p - \frac{1}{p}\right) dp = \int dy + c.$$

Hence,

$y = \frac{p^2}{2} - \log\, p + c$, where c is an arbitrary constant.

Here,

it is difficult to eliminate p.

Therefore,

general solution can be obtained by taking $x = p + \frac{1}{p}$; $y = \frac{p^2}{2} - \log\, p + c$.

EXERCISES:

1. Solve $y = (1 + p)x + p^2$.

2. Solve $xp - y + \sqrt{x}$.

3. Solve $y = 2p + 3p^2$.

4. Solve $y + px = p^2x^4$.

5. Solve $y^2p^2 - 3xp + y = 0$.

4.5 PARTIAL DIFFERENTIAL EQUATIONS:

Definition-4.5.1:

When a differential equation contains one or more partial derivatives of an unknown function of two or more variables (independent); then it is called a $partial\ differential\ equation$.

Note:

1. We consider, generally x and y as independent variables and z as dependent variable.

   (ie)., $z = f(x, y)$.

2. We denote partial derivatives of first and higher orders as

$$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^2 z}{\partial x \partial y}$$

   For which symbols p, q, r, t, s respectively will be used in this topic.

   (ie).,

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = s, \quad \frac{\partial^2 z}{\partial y^2} = t$$

Definition-4.5.2:

The $order$ of a partial differential equation is defined as the order of the highest partial derivative occurring in it and $the\ degree$ is defined as the exponent of the highest order partial derivative.

Definition-4.5.3:

A partial differential equation is said to be *linear* if the dependent variable and its partial derivatives occur only in first degree and are not multiplied together in the differential equation. Otherwise, the equation is called $Non-linear$ differential equation.

Examples:

1. $\frac{\partial z}{\partial x} - 5\frac{\partial z}{\partial y} = 2z + \sin(x - 2y)$;

   order = 1, degree = 1 and it is linear.

2. $xz\frac{\partial z}{\partial x} + yz\frac{\partial z}{\partial y} = 3xy$;

   order = 1, degree = 1 and it is non-linear.

3. $(x^2 - z^2)\frac{\partial z}{\partial x} + yz\frac{\partial z}{\partial y} = -3xy$;

   order = 1, degree = 1 and it is non-linear.

4. $5\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} + 3\frac{\partial^2 z}{\partial y^2} = 0$;

   order = 2, degree = 1 and it is linear.

5. $z\frac{\partial z}{\partial x} + 5\frac{\partial z}{\partial y} = 2y$;

   order = 1, degree = 1 and it is non-linear.

4.6 FORMATIONS AND SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS:

In general, there are two ways for derivation of partial differential equations.

(i) By eliminating arbitrary constants from the given relation between variables.

(ii) By eliminating arbitrary functions from the given relation between variables.

SOLVED PROBLEMS:

Problem 1.

Form partial differential equation by eliminating arbitrary constants from the relation

$z = (2x + a)(2y + b)$.

Solution:

Given $z = (2x + a)(2y + b)$ ……………………..(1)

Differentiate partially w.r.to x and w.r.to y,

we get,

$$\frac{\partial z}{\partial x} = (2y + b)(2 + 0) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

and $\frac{\partial z}{\partial x} = (2x + a)(2 + 0)$ ……………………………… (3)

Multiply (2) and (3); we get,

$$\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = (2y + b)(2x + a)(2)$$

$$\Rightarrow \quad pq = 4z,$$

which is the required partial differential equation.

Problem 2.

Form partial differential equation by eliminating arbitrary constants from the relation

$z = ax + by + a^3 + b^3$.

Solution:

Given $z = ax + by + a^3 + b^3$ ……………………..(1)

Differentiate partially w.r.to x and w.r.to y,

we get,

$\frac{\partial z}{\partial x} = a$ ……………………………… (2)

and $\frac{\partial z}{\partial x} = b$ ……………………………… (3)

Put these values of a and b in (1); we get,

$$z = x\frac{\partial z}{\partial x} + y\,\frac{\partial z}{\partial y} + \left(\frac{\partial z}{\partial x}\right)^3 + \left(\frac{\partial z}{\partial y}\right)^3.$$

$$\Rightarrow \quad z = px + qy + (p)^3 + (q)^3,$$

which is the required partial differential equation.

EXERCISES:

1. Form partial differential equation by eliminating arbitrary constants from the relation $z = ax + by + ab$.

2. Form partial differential equation by eliminating arbitrary constants from the relation $z = ax + (2 - a)y + b$.

3. Form partial differential equation by eliminating arbitrary constants from the relation $z = \frac{1}{3}ax^3 + \frac{1}{3}ay^3$.

4. Form partial differential equation by eliminating arbitrary constants from the relation $z = ax + a^2y^2 + b^2$.

4.7 STANDARD FORM $\boldsymbol{Pp + Qq = R}$.

Definition-4.7.1 (***Lagrange's Linear Equation***):

A partial differential equation of the form $Pp + Qq = R$ where P, Q, R are functions of x, y, z (which is of first order and linear in p and q) is known as *Lagrange's Linear Equation.*

Working Method to solve $Pp + Qq = R$:

Step 1:

Write given equation in the form of $Pp + Qq = R$ and find P, Q, R.

Step 2:

Write Lagrange's auxiliary equations.

Step 3:

Find two independent solutions $u = a$ and $v = b$ of equations given in step-2.

Step 4:

Write $f(u, v) = 0$, which is general solution (integral) of the given equation.

SOLVED PROBLEMS:

Problem 1.

Solve $p + q = \cos x$.

Solution:

Given $p + q = \cos x$. ……………………(1)

Compare it with $Pp + Qq = R$.

Here $P = 1, Q = 1, R = \cos x$.

∴ Auxiliary (subsidiary) equation are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{\cos x} \quad \text{..........................(2)}$$

Taking first two members of (2), we get,

$$dx = dy$$

Integrating, $x = y + a$ or $x - y = a$ ……………………………(3)

Now,

Taking first and last members of (2), we get,

$$\cos x \; dx = dz$$

Integrating, $\sin x = z + b$ or $\sin x - z = b$ ……………………………(4)

Thus from (3) and (4), we get,

$$u = a \text{ and } v = b,$$

where $u(x, y, z) = x - y$, $v(x, y, z) = \sin x - z$.

∴ The general solution of given equation is

$$f(x - y, \sin x - z) = 0$$

(or) $\sin x - z = f(x - y)$, where f is any arbitrary function

EXERCISES:

1. Solve $\alpha p + \beta q = \gamma$.

2. Solve $6p + 7q = 8$.

3. Solve $pz = x$.

4. Solve $px + qq = 5z$.

5. Solve $x^2p + y^2q = z^2$.

6. Solve $yzp + zxq = xy$.

Unit-V:

Laplace transformation – Inverse Laplace transform.

5.1 LAPLACE TRANSFORMATION:

Introduction:

In this chapter we introduce the concept of Laplace transform of a function $f(x)$ and see how it is useful to solve certain types of differential equations. We note that the operation of differentiation transforms a function $f(x)$ onto its derivative $f'(x)$. If we use D to denote differentiation then the transformation can be written as $D[f(x)] = f'(x)$.

Similarly, integration can be thought of a transformation given by

$$I[f(x)] = \int_0^x f(t)dt$$

Both these transformations operate on functions to produce another function. A general transformation T of functions is said to be *linear* if

$$T[\alpha f(x) + \beta g(x)] = \alpha T[f(x)] + \beta T[g(x)]$$

for all admissible functions f, g and scalars α, β. We notice that differentiation and integration are linear operators.

We now introduce another operator on functions known as $Laplace\ transform$.

Definition-5.1.1:

Let $f(x)$ be a function defined on$[0, \infty)$. The $\boldsymbol{Laplace\ transform\ L}$ is defined by

$$L[f(x)] = \int_0^\infty e^{-sx} f(x)dx = F(s)$$

The Laplace transform L acts on any function $f(x)$ for which the above integral exists.

Note-1:

L is a linear operator.

Proof:

For,

$$T[\alpha f(x) + \beta g(x)] = \int_0^\infty e^{-sx}[\alpha f(x) + \beta g(x)]dx$$

$$= \alpha \int_0^\infty e^{-sx} f(x)dx + \beta \int_0^\infty e^{-sx} g(x)dx$$

$$= \alpha L[f(x)] + \beta L[g(x)]$$

Note-2:

$L[f(ax)] = \frac{1}{a} F(\frac{s}{a})$.

Proof:

By definition,

$$L[f(x)] = \int_0^\infty e^{-sx} f(x)dx = F(s)$$

$$\therefore \quad L[f(ax)] = \int_0^\infty e^{-sx} f(ax)dx$$

Put $ax = y$.

Hence,

$$dx = \frac{dy}{a}.$$

$$\therefore \quad L[f(ax)] = \frac{1}{a} \int_0^\infty e^{-\left(\frac{s}{a}\right)y} f(y)dy$$

$$= \frac{1}{a} F(\frac{s}{a}).$$

SOLVED PROBLEMS:

Problem-1.

Prove that $L(x^n) = \frac{\Gamma(n+1)}{s^{n+1}}$.

Proof:

$$L[x^n] = \int_0^\infty e^{-sx} x^n dx$$

Put $sx = y$.

Hence,

$$dx = \frac{dy}{s}.$$

$$\therefore \quad L[x^n] = \frac{1}{a}\int_0^\infty e^{-y}\left(\frac{y}{s}\right)^n\left(\frac{dy}{s}\right)dy$$

$$= \frac{1}{s^{n+1}}\int_0^\infty y^n e^{-y} dy.$$

$$= \frac{\Gamma(n+1)}{s^{n+1}}. \text{ (by definition of Gamma function)}$$

Result:

1. $L[x^n] = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$ (where n is a positive integer)
2. $L[1] = \frac{1}{s}$
3. $L[x] = \frac{1}{s^2}$
4. $L[x^2] = \frac{2}{s^3}$
5. $L(\sqrt{x}) = \frac{\Gamma\left(\frac{3}{2}\right)}{s^{3/2}}$

$$= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{3/2}}$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}.$$

Problem-2.

Prove that $L[e^{ax}] = \frac{1}{s-a}$ if $s - a > 0$.

Proof:

$$L[e^{ax}] = \int_0^\infty e^{-sx} e^{ax} dx$$

$$= \int_0^\infty e^{-(s-a)x} dx$$

$$= \lim_{m\to\infty} \int_0^\infty e^{-(s-a)x} dx$$

$$= \left[-\frac{e^{-(s-a)x}}{(s-a)} \right]_0^m$$

$$= \left[-\frac{e^{-(s-a)m}}{(s-a)} + \frac{1}{s-a} \right]$$

$$= \frac{1}{s-a} \text{ if } s - a > 0.$$

Problem-3.

Show that $L[\cos\ ax] = \frac{s}{s^2+a^2}$

Proof:

$$L[\cos\ ax] = \text{Real part of } \int_0^\infty e^{-ax} e^{i\,ax} dx$$

$$= \text{Real part of } L(e^{i\,ax})$$

$$= \text{Real part of } \left(\frac{1}{s-ai}\right)$$

$$= \text{Real part of } \left(\frac{s+ai}{s^2+a^2}\right)$$

$$= \frac{s}{s^2+a^2}.$$

Problem-4.

Show that $L[\cos\ hax] = \frac{s}{s^2-a^2}$

Proof:

$$L[\cos\ hax] = L\left(\frac{e^{ax}+e^{-ax}}{2}\right)$$

$$= \frac{1}{2}L(e^{ax}) + \frac{1}{2}L(e^{-ax})$$

$$= \frac{1}{2(s-a)} + \frac{1}{2(s+a)}$$

$$= \frac{s}{s^2-a^2}.$$

Problem-5.

Show that $L[f'(x)] = sL[f(x)] - f(0)$.

Proof:

$$L[f'(x)] = \int_0^\infty e^{-sx} f'(x)dx$$

$$= \left\{[f(x)e^{-sm}]_0^m - \int_0^m f(x)(-s)e^{-sx}dx\right\}$$

$$= \left\{[f(x)e^{-sm} - f(0)] - s\int_0^m e^{-sx}f(x)dx\right\}$$

$$= -f(0) + s\int_0^m e^{-sx}f(x)dx$$

$$= sL[f(x)] - f(0).$$

Problem-6.

Show that $L[f''(x)] = s^2L[f(x)] - sf(0) - f'(0)$.

Proof:

$$L[f''(x)] = L[g'(x)] \text{ where } g(x) = f'(x)$$

$$= sL[g(x)] - g(0) \text{ (by previous problem)}$$

$$= sL[f'(x)] - f'(0)$$

$$= s[sL[f(x)] - f(0)] - f'(0)$$

$$= s^2L[f(x)] - sf(0) - f'(0).$$

Note:

In general,

$$L\left[f^{(n)}(x)\right] = s^nL[f(x)] - s^{n-1}f(0) - s^{n-1}f'(0) - s^{n-2}f''(0) - \cdots - f^{(n-1)}(0).$$

Problem-7.

Find the Laplace transform of $t^2 + \cos\ 2t \cos\ t + sin^2\ 2t$.

Solution:

$$L[t^2 + \cos\ 2t \cos\ t + sin^2\ 2t] = L(t^2) + L(\cos\ 2t \cos\ t) + L(sin^2\ 2t)$$

$$= \frac{2}{s^3} + L\left[\frac{1}{2}(\cos\ 3t + \cos\ t)\right] + L\left[\frac{1}{2}(1 - \cos\ 4t)\right]$$

$$= \frac{2}{s^3} + \frac{1}{2}[L(\cos\ 3t) + L(\cos\ t)] + \frac{1}{2}[L(1) - L(\cos\ 4t)]$$

$$= \frac{2}{s^3} + \frac{1}{2}[L(\cos\ 3t) + L(\cos\ t)] + \frac{1}{2}[L(1) - L(\cos\ 4t)]$$

$$= \frac{2}{s^3} + \frac{1}{2}\left[\frac{s}{s^2+9} + \frac{s}{s^2+1}\right] + \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+16}\right]$$

Problem-8.

Find the Laplace transform of $xe^{-x} \cos\ x$.

Solution:

$$L(xe^{-x}\cos\ x) = -\frac{d}{ds}L(e^{-x}\cos\ x)$$

$$= -\frac{d}{ds}F(s+1),$$

where $F(s) = L(\cos\ x) = \frac{s}{s^2+1}$.

Now,

$$F(s+1) = \frac{s+1}{(s+1)^2+1}$$

$$= \frac{s+1}{s^2+2s+1}$$

$$\therefore\quad L(xe^{-x}\cos\ x) = -\frac{d}{ds}\left(\frac{s+1}{s^2+2s+1}\right)$$

$$= \frac{s^2+2s}{(s^2+2s+2)^2}.\quad \text{(verify)}$$

Problem-9.

Find the Laplace transform of $x^2 \cos h\ ax$.

Solution:

$$L(x^2 \cos h\ ax) = \frac{d^2}{ds^2}L(\cos h\ ax)$$

$$= \frac{d^2}{ds^2}\left(\frac{s}{s^2-a^2}\right)$$

$$= -\frac{d}{ds}\left[\frac{a^2+s^2}{(s^2-a^2)^2}\right]$$

$$= \frac{2s(s^2+3a^2}{(s^2-a^2)^3}$$

Problem-10.

Find the Laplace transform of x^2e^{-ax}.

Solution:

$$L[x^2e^{-ax}] = (-1)^2\frac{d^2}{ds^2}[L(e^{-ax})]$$

$$= \frac{d^2}{ds^2}\left(\frac{1}{s+a}\right)$$

$$= \frac{d}{ds}\left[\frac{-1}{(s+a)^2}\right]$$

$$= \frac{2}{(s+a)^3}.$$

EXERCISES:

1. Prove that $L[e^{-ax}] = \frac{1}{s+a}$ if $s + a > 0$.
2. Show that $L[\sin\ ax] = \frac{a}{s^2+a^2}$.
3. Show that $L[\sin\ hax] = \frac{a}{s^2-a^2}$
4. Find the Laplace transform of x^2e^{-4x}.
5. Find the Laplace transform of $x^2 \sin\ 2x$.
6. Find the Laplace transform of $x \cos\ 2x$.

5.2 INVERSE LAPLACE TRANSFORM:

Definition-5.2.1:

When $f(x)$ is continuous and $L[f(x)] = F(s)$, we have $L^{-1}[F(s)] = f(x)$ and L^{-1} is called the ***inverse laplace transform*** and $f(x)$ is called the ***inverse laplace transform*** of $F(s)$.

Results:

1. $L^{-1}\left[\frac{1}{s}\right] = 1$
2. $L^{-1}\left[\frac{1}{s^2}\right] = x$
3. $L^{-1}\left[\frac{1}{s-a}\right] = e^{ax}$
4. $L^{-1}\left[\frac{1}{s+a}\right] = e^{-ax}$
5. $L^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin\ ax$
6. $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos\ ax$
7. $L^{-1}\left[\frac{a}{s^2-a^2}\right] = \sin h\ ax$
8. $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cos h\ ax$
9. $L^{-1}\left[\frac{1}{(s-a)^2}\right] = xe^{ax}$

Note:

Since L is linear, L^{-1} is also linear.

Problem-1.

Show that $L^{-1}[F(s+a)] = e^{-ax}L^{-1}[F(s)]$.

Proof:

Let $L[f(x)] = F(s)$.

Then,

we know that, $L[e^{-ax}f(x)] = F(s+a)$.

Hence,

$$L^{-1}[F(s+a)] = e^{-ax}f(x) = e^{-ax}L^{-1}[F(s)].$$

Problem-2.

Show that $L^{-1}[F(\lambda s)] = \frac{1}{\lambda}F\left(\frac{x}{\lambda}\right)$.

Proof:

We know that, $L[f(ax)] = \frac{1}{a}F\left(\frac{s}{a}\right)$.

$$L^{-1}\left[\frac{1}{a}F\left(\frac{s}{a}\right)\right] = f(ax).$$

Put $\frac{1}{a} = \lambda$.

Then, we have $L^{-1}[\lambda F(\lambda s)] = F\left(\frac{x}{\lambda}\right)$.

Hence,

$$L^{-1}[F(\lambda s)] = \frac{1}{\lambda}F\left(\frac{x}{\lambda}\right).$$

Problem-3.

Show that $L^{-1}[F'(s)] = -xL^{-1}[F(s)]$.

Proof:

We know $L[xf(x)] = -F'(s)$.

$$\therefore \; L^{-1}[F'(s)] = -xf(x)$$

$$= -xL^{-1}[F(s)].$$

Problem-4.

Find the inverse Laplace transform of $\frac{1}{(s+3)^2+25}$.

Solution:

$$L^{-1}\left[\frac{1}{(s+3)^2+25}\right] = e^{-3x}L^{-1}\left[\frac{1}{s^2+5^2}\right]$$

$$= \frac{1}{5}e^{-3x}L^{-1}\left[\frac{5}{s^2+5^2}\right]$$

$$= \frac{1}{5}e^{-3x}\sin\ 5x$$

Problem-5.

Find the inverse Laplace transform of $\frac{s}{(s+2)^2}$.

Solution:

$$L^{-1}\left[\frac{s}{(s+2)^2}\right] = L^{-1}\left[\frac{s+2-2}{(s+2)^2}\right]$$

$$= L^{-1}\left[\frac{1}{s+2}\right] - 2L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$= e^{-2x}\left[\frac{1}{s}\right] - 2e^{-2x}L^{-1}\left[\frac{1}{s^2}\right]$$

$$= e^{-2x} - 2e^{-2x}x$$

$$= e^{-2x}(1-2x).$$

Problem-6.

Find the inverse Laplace transform of $\frac{s+1}{s^2+2s+2}$.

Solution:

$$L^{-1}\left[\frac{s+1}{s^2+2s+2}\right] = L^{-1}\left[\frac{s+1}{(s+1)^2+1}\right]$$

$$= e^{-x}L^{-1}\left[\frac{s}{s^2+1}\right]$$

$$= e^{-x}\cos\ x.$$

Problem-7.

Find the inverse Laplace transform of $\frac{s}{a^2s^2+b^2}$.

Solution:

$$\frac{s}{a^2s^2+b^2} = \frac{s}{a^2\left[s^2+\left(\frac{b}{a}\right)^2\right]}$$

$$L^{-1}\left[\frac{s}{a^2s^2+b^2}\right] = \frac{1}{a^2}L^{-1}\left[\frac{s}{s^2+\left(\frac{b}{a}\right)^2}\right]$$

$$= \frac{1}{a^2}\cos\left(\frac{bx}{a}\right)$$

Problem-8.

Find $L^{-1}\left(\frac{1}{s(s+1)(s+2)}\right)$.

Solution:

Let $\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{(s+2)}$.

We can prove $A = \frac{1}{2}, B = -1$ and $C = \frac{1}{2}$.

$\therefore \quad \frac{1}{s(s+1)(s+2)} = \frac{1}{2s} - \frac{1}{(s+1)} + \frac{1}{2(s+2)}$

Hence, $L^{-1}\left(\frac{1}{s(s+1)(s+2)}\right) = \frac{1}{2}L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{2}L^{-1}\left(\frac{1}{s+2}\right)$

$$= \frac{1}{2}(1) - e^{-x}L^{-1}\left(\frac{1}{s}\right) + \frac{1}{2}e^{-2x}L^{-1}\left(\frac{1}{s}\right)$$

$$= \frac{1}{2} - e^{-x} + \frac{1}{2}e^{-2x}$$

EXERCISES:

1. Find the inverse Laplace transform of $\frac{s+3}{s^2+2s+10}$.
2. Find the invers Laplace transform of $\frac{1}{s(s+a)}$.
3. Find the invers Laplace transform of $\frac{s}{(s^2-1)^2}$.
4. Find the invers Laplace transform of $\frac{s+a}{(s^2+4s+5)^2}$
5. Find $L^{-1}\left(\frac{s^2-s+2}{s(s-3)(s+2)}\right)$.

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